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Uncertainty relations in curved spaces

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Abstract
Uncertainty relations for particle motion in curved spaces are discussed. The relations are shown to be topologically invariant. A new coordinate system on a sphere appropriate to the problem is proposed. The case of a sphere is considered in detail. The investigation can be of interest for string and brane theory, solid state physics (quantum wires) and quantum optics.

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1. Introduction

The Heisenberg uncertainty relation

$$\Delta x \cdot \Delta p_x \geqslant \frac{\hbar}{2} \tag{1}$$

holds for quantum motion on a plane (see, for example, [1]); here Δx, Δp_x are coordinate and momentum dispersions, respectively. This inequality can be derived from the well-known relation $[\hat{x}, \hat{p}_x] = \mathrm{i}\hbar$. However, in quantum mechanics on a circle one has the standard commutational relation for coordinate $\hat{\varphi}$, with $\varphi \in [0, 2\pi)$, and momentum $\hat{p}_\varphi$, but the uncertainty relation cannot be stronger than

$$\Delta\varphi \cdot \Delta p_\varphi \geqslant 0. \tag{2}$$

The momentum dispersion can be equal to zero while the coordinate one never becomes infinite. In contrast to (1), inequality (2) *is not informative at all,* since a product of two non-negative values cannot be negative. We have to mention here that in this paper an inequality of this type will be regarded as an uncertainty relation only if it contains a number in its rhs without dependence on the wavefunction. If we neglect this requirement, stronger relations are possible (see, for example, section 2) but they would be extremely sensitive to the choice of the coordinate system.

The problem holds, of course, for any compact manifold. For the sphere this problem is even more complicated because of the absence of a self-adjoint momentum operator related to the azimuth angle (due to boundary terms in matrix elements caused by the coordinate system poles). We propose a solution of the problem for any coordinates with closed coordinate lines.

We do not consider the peculiar properties of quantum mechanics connected with extrinsic and intrinsic geometries; they add an additional potential to the Hamiltonian [2].

The foundations of quantum mechanics (see, for example, [3]) and, in particular, uncertainty relations [4, 5] have been the focus of unprecedented activity. Some subtle mathematical points are taken into account, such as self-adjointness of operators, their domains and so on [4, 6], which are usually disregarded in physical papers.

Modern physics often encounters quantum motion in curved spaces. In nanoelectronics the quantum wires not only may be curved or closed, but sometimes fail even to be manifolds at all (triple vertex, etc [7, 8]). Superstring theory [9] considers strings and branes with different topologies. And, of course, our physical space is obviously curved. Similar problems arise in quantum optics (photon number and phase operators), but the main issue there is the correct definition of the phase operator.

The cases of a circle and an arbitrary curved line are presented in section 2. Section 3 is devoted to quantum mechanics on the sphere; stereographic coordinates and spaces diffeomorphic to the plane are considered; uncertainty relations are shown to be the usual ones. In section 4 we apply a method of stereographic projection to the circle. New coordinates on the sphere, appropriate to the method of section 2, are proposed in section 5. In section 6 and in the appendix we look at arbitrary manifolds with closed coordinate lines; the diffeomorphic invariance of uncertainty relations is shown. The phase space structure is discussed in section 7.

2. One-dimensional manifold

For the circle the situation described above arises because the operator $\hat{\varphi}$ takes out physical states from the Hilbert space of 2π-periodical functions [4, 6]. Hence a product of operators $\hat{\varphi}$ and $\hat{p}_\varphi$ is not well defined and one should be very careful while working with these operators. It is easy to show that for any normalized state $|\Psi\rangle$ the following inequality is valid:

$$\Delta\varphi \cdot \Delta p_\varphi \geqslant |\mathrm{Im}\langle\hat{\varphi}\Psi|\hat{p}_\varphi\Psi\rangle|. \tag{3}$$

Indeed, the definition of a dispersion for any observable reads [1, 4, 6, 10]: $(\Delta\varphi)^2 = \langle(\hat{\varphi}-\bar{\varphi})\Psi|(\hat{\varphi}-\bar{\varphi})\Psi\rangle$, where $\bar{\varphi}$ is a mean value of the observable. Using the Cauchy inequality $|\langle\Psi_1|\Psi_2\rangle| \leqslant |\Psi_1|\cdot|\Psi_2|$, one gets [4, 11]:

$$\Delta\varphi\cdot\Delta p_\varphi \geqslant |\langle(\hat{\varphi}-\bar{\varphi})\Psi|(\hat{p}_\varphi-\bar{p}_\varphi)\Psi\rangle| \geqslant |\mathrm{Im}\langle(\hat{\varphi}-\bar{\varphi})\Psi|(\hat{p}_\varphi-\bar{p}_\varphi)\Psi\rangle| = |\mathrm{Im}\langle\hat{\varphi}\Psi|\hat{p}_\varphi\Psi\rangle|.$$

In this derivation we did not use a product of coordinate and momentum operators. If the product is well defined for any order of operators in it, then the rhs of (3) turns out to be $\frac{1}{2}\langle\Psi|[\hat{\varphi},\hat{p}_\varphi]\Psi\rangle$, which is the classical result.

On the circle one cannot use the commutator of $\hat{\varphi}$ and $\hat{p}_\varphi$, but integration by parts and the $\langle\Psi|\Psi\rangle = 1$ condition yield

$$\begin{aligned}\mathrm{Im}\langle\hat{\varphi}\Psi|\hat{p}_\varphi\Psi\rangle &= \mathrm{Im}\frac{\hbar}{\mathrm{i}}\int_0^{2\pi}\mathrm{d}\varphi\,\Psi^*(\varphi)\varphi\frac{\partial}{\partial\varphi}\Psi(\varphi)\\ &= -\frac{\hbar}{2}\int_0^{2\pi}\mathrm{d}\varphi\,\varphi\frac{\partial}{\partial\varphi}|\Psi|^2 = \frac{\hbar}{2}(1-2\pi|\Psi(2\pi)|^2) = \frac{\hbar}{2}(1-2\pi|\Psi(0)|^2).\end{aligned}$$

In general, if one rotates the coordinate system $\varphi \to \varphi + \delta \bmod 2\pi$ the rhs of the obtained formula changes for a given Ψ. For $\Psi_k(\varphi) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{\mathrm{i}k\varphi}{\hbar}\right)$ (here and hereafter k is an integer number times $\hbar$) the rhs of (3) equals zero, hence we have relation (2). It is the strongest possible relation with a pure number on its rhs.

This result can be obtained in another way. Let us consider a periodic coordinate operator $\hat{\tilde{\varphi}} = f(\varphi)$, demanding that $f(\varphi) = \varphi \bmod 2\pi$ and $0 \leqslant f(\varphi) < 2\pi$ with the variable $\varphi \in (-\infty, \infty)$. It defines the function properly and we have

$$[\hat{\tilde{\varphi}}, \hat{p}_\varphi] = \mathrm{i}\hbar \left(1 - \sum_{n=-\infty}^{\infty} 2\pi \delta(\varphi - 2\pi n)\right). \tag{4}$$

The scalar product is $\langle \Psi_1 | \Psi_2 \rangle = \int_{0-\epsilon}^{2\pi-\epsilon} \Psi_1^* \Psi_2 \, \mathrm{d}\varphi$. Only singularities at $\varphi = 0$ play a role while those at $\varphi = 2\pi$ are disregarded. The standard relation

$$\Delta\varphi \cdot \Delta p_\varphi \geqslant \tfrac{1}{2} \langle \Psi | [\hat{\tilde{\varphi}}, \hat{p}_\varphi] \Psi \rangle$$

and the inequality (3) give the same result (2).

All this is valid for an arbitrary closed curve with a coordinate $\varphi \in [0, A)$ and a length element $\mathrm{d}l = h(\varphi)\, \mathrm{d}\varphi$. We have to consider a self-adjoint operator

$$\hat{p}_\varphi = \frac{\hbar}{\mathrm{i}} \frac{1}{\sqrt{h(\varphi)}} \frac{\partial}{\partial \varphi} \sqrt{h(\varphi)}$$

and a scalar product

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{0-\epsilon}^{A-\epsilon} \Psi_1^* \Psi_2 h(\varphi) \, \mathrm{d}\varphi.$$

One can refer to all the integrals as the ordinary ones and the rhs of (3) can be integrated by parts. The δ-function approach (4) can also be used. It yields again the previous result (2). The inequality is saturated by functions

$$\frac{1}{A\sqrt{h(\varphi)}} \exp\left(\frac{2\mathrm{i}\pi k \varphi}{A\hbar}\right).$$

So, the uncertainty relation (2) is invariant under any smooth deformation of the circle (topological invariance). We shall show that this property is valid not only for the one-dimensional case.

Note that due to the periodicity of all the functions one can also consider only the upper limit singularities instead of the lower limit ones or take both with the factor $\frac{1}{2}$ for each one.

3. Stereographic projection

As we show later, the methods of section 2 can be applied to an arbitrary closed coordinate line on any manifold. But for the azimuth angle of spherical coordinates it does not work because the momentum operator is not self-adjoint (due to the boundary terms in coordinate poles).

However in this case one may consider a stereographic projection. It allows us to introduce new operators on the projection plane:

$$\begin{cases} \hat{q}_1 = 2R \cot\left(\dfrac{\vartheta}{2}\right) \cos\varphi \\ \hat{p}_1 = \dfrac{\mathrm{i}\hbar}{R} \left(\dfrac{\sin\varphi}{\sin\vartheta} \dfrac{\partial}{\partial\varphi} + \cos\varphi \dfrac{\partial}{\partial\vartheta}\right) \sin^2 \dfrac{\vartheta}{2} \end{cases} \tag{5}$$

$$\begin{cases} \hat{q}_2 = 2R\cot\left(\frac{\vartheta}{2}\right)\sin\varphi \\ \hat{p}_2 = \frac{\mathrm{i}\hbar}{R}\left(-\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\vartheta}\right)\sin^2\frac{\vartheta}{2} \end{cases} \tag{6}$$

with the length element $\mathrm{d}l^2 = R^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) = \sin^4\frac{\vartheta}{2}\left(\mathrm{d}q_1^2 + \mathrm{d}q_2^2\right)$ where $\sin^2\frac{\vartheta}{2} = \left(1 + \frac{q_1^2+q_2^2}{4R^2}\right)^{-1}$, the surface area element $\mathrm{d}S = R^2\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \sin^4\frac{\vartheta}{2}\,\mathrm{d}q_1\,\mathrm{d}q_2$ and the scalar product $\langle\Psi_1|\Psi_2\rangle = \int\Psi_1^*\Psi_2\,\mathrm{d}S$. Self-adjoint momentum operators on the sphere are given by

$$\frac{\hbar}{\mathrm{i}}\frac{1}{\sin^2\frac{\vartheta}{2}}\frac{\partial}{\partial q_k}\sin^2\frac{\vartheta}{2}$$

after rewriting the derivatives in terms of the spherical angles. These operators are suggested by the projection plane description.

Any differential operator can be written in terms of new coordinates and momenta (5), (6). For example, the free particle kinetic energy is

$$-\frac{\hbar^2}{2}\Delta = \left(1 + \frac{\hat{q}_1^2 + \hat{q}_2^2}{4R^2}\right)\frac{\left(\hat{p}_1^2 + \hat{p}_2^2\right)}{2}\left(1 + \frac{\hat{q}_1^2 + \hat{q}_2^2}{4R^2}\right).$$

There are no periodic coordinates in this system and one has the standard commutational and uncertainty relations

$$[\hat{q}_1, \hat{p}_1] = \mathrm{i}\hbar \qquad [\hat{q}_2, \hat{p}_2] = \mathrm{i}\hbar \qquad [\hat{p}_1, \hat{p}_2] = 0$$
$$\Delta q_1 \cdot \Delta p_1 \geqslant \frac{\hbar}{2} \qquad \Delta q_2 \cdot \Delta p_2 \geqslant \frac{\hbar}{2}.$$

The latter can be obtained using (3) because the commutators are well defined at a dense subset of physical wavefunctions.

Ordinary quantum mechanics on the plane is different due to the fact that the area element $\mathrm{d}S = \mathrm{d}x_1\,\mathrm{d}x_2$ does not have any factor tending to zero at the coordinate infinity which is drastic, of course, for functions from the Hilbert space. Spherical infinity (the north pole) is an ordinary point where a wavefunction can take any value while at the plane infinity wavefunctions must go to zero fast enough. It should also be mentioned that spherical coordinates on the sphere are related to polar ones on the plane by the stereographic projection; and polar coordinates have the same problem: the radial momentum operator is not self-adjoint.

Note that all our formulae do not depend on the particular area element. If one has a diffeomorphic image of the plane then the coordinate system x_1, x_2 can be moved to it, possibly with a different area element, but with infinite points mapping onto infinite points. Relation (1) still holds, since the particular area element is irrelevant for our derivation. In other words, relation (1) is invariant under any smooth deformation of the plane. Obviously, one can establish it also for an arbitrary manifold diffeomorphic to the sphere by generalizing the operators (5) and (6). But one has to bear in mind that these coordinates can reach infinite values in spite of the finite total volume of the manifold. Another shortcoming is that some physical states are excluded from the coordinate operator domains. We discuss this in more detail in the next section.

Note also that the stereographic projection is valid for an arbitrary dimension. For S^n the length element is

$$\mathrm{d}l^2 = \sin^4\frac{\vartheta}{2}\left(\mathrm{d}q_1^2 + \mathrm{d}q_2^2 + \cdots + \mathrm{d}q_n^2\right)$$

the surface element $\mathrm{d}S = \sin^{2n}\frac{\vartheta}{2}\,\mathrm{d}q_1\,\mathrm{d}q_2\cdots\mathrm{d}q_n$, the momenta

$$\hat{p}_i = \frac{\hbar}{\mathrm{i}}\frac{1}{\sin^n\frac{\vartheta}{2}}\frac{\partial}{\partial q_i}\sin^n\frac{\vartheta}{2}$$

the commutational and the uncertainty relations are standard. The free particle kinetic energy is

$$-\frac{\hbar^2}{2}\Delta = \frac{1}{2}\left(1+\frac{\hat{q}_1^2+\cdots+\hat{q}_n^2}{4R^2}\right)^{n/2} \times \sum_{i=1}^{n}\left(\hat{p}_i\left(1+\frac{\hat{q}_1^2+\cdots+\hat{q}_n^2}{4R^2}\right)^{4-2n}\hat{p}_i\right)\left(1+\frac{\hat{q}_1^2+\cdots+\hat{q}_n^2}{4R^2}\right)^{n/2}.$$

4. Stereographic projection for the circle

In section 3 we obtained the standard relations (1) for the quantum motion on any n-dimensional sphere, $n \geqslant 2$. In the same manner one can get (1) for $n = 1$, but earlier we had (2) for the circle. What is the matter? To elucidate why we had different results for the circle, let us consider a stereographic projection from the circle onto the line with zero angle being at the contact point and $\varphi \in [-\pi, \pi)$. Then $x = 2R\tan\varphi/2$. Thus we use in this section the $[-\pi, \pi)$ interval instead of the $[0, 2\pi)$ one. This trick changes the operator dispersions [12] but, as can be easily seen, leaves relation (2) the same.

We start from the following generalization:

$$x \to \tilde{x} = \frac{2R}{\alpha}\tan\frac{\alpha\varphi}{2} \qquad \alpha \in [0, 1].$$

The length element is $\mathrm{d}l = R\,\mathrm{d}\varphi = \cos^2\frac{\alpha\varphi}{2}\,\mathrm{d}\tilde{x}$ and the momentum

$$\hat{p}_{\tilde{x}} = \frac{\hbar}{\mathrm{i}R}\cos\frac{\alpha\varphi}{2}\frac{\partial}{\partial\varphi}\cos\frac{\alpha\varphi}{2}$$

(its self-adjointness is a matter of direct calculation). The $\tilde{x}$ domain is $\left[-\frac{2R}{\alpha}\tan\frac{\alpha\pi}{2}, \frac{2R}{\alpha}\tan\frac{\alpha\pi}{2}\right)$. If $\alpha \to 0$ one has $\tilde{x} \to R\varphi$ and $\hat{p}_{\tilde{x}} \to \frac{\hbar}{\mathrm{i}R}\frac{\partial}{\partial\varphi}$, i.e. ordinary quantum mechanics on the circle with the angle variable φ and the uncertainty relation (2). If $\alpha \to 1$ then $\tilde{x} \to x \in \mathbb{R}^1$ while

$$\hat{p}_{\tilde{x}} \to \frac{\hbar}{\mathrm{i}R}\cos\frac{\varphi}{2}\frac{\partial}{\partial\varphi}\cos\frac{\varphi}{2}$$

so the standard relation (1) is valid for $\tilde{x}$ and $p_{\tilde{x}}$. And what about arbitrary α? A simple calculation yields for the rhs of (3)

$$\begin{aligned}\mathrm{Im}\langle\hat{\tilde{x}}\Psi|\hat{p}_{\tilde{x}}\Psi\rangle &= \mathrm{Im}\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\frac{2R}{\alpha}\tan\frac{\alpha\varphi}{2}\Psi^*(\varphi)\frac{\hbar}{\mathrm{i}R}\cos\frac{\alpha\varphi}{2}\frac{\partial}{\partial\varphi}\cos\frac{\alpha\varphi}{2}\Psi(\varphi)\\ &= \frac{\hbar}{2}\left(1-\frac{\sin\alpha\varphi}{\alpha}|\Psi(\varphi)|^2\Big|_{-\pi}^{\pi}\right) = \frac{\hbar}{2}\left(1-\frac{2\sin\alpha\pi}{\alpha}|\Psi(-\pi)|^2\right) \qquad (7)\end{aligned}$$

where the periodicity $\Psi(-\pi) = \Psi(\pi)$ and the normalization condition $\langle\Psi|\Psi\rangle = 1$ were taken into account. For any $\alpha < 1$ the uncertainty relation has the form of (2): $\Delta\tilde{x}\cdot\Delta p_{\tilde{x}} \geqslant 0$ (it is saturated by any function with $|\Psi(\pi)|^2 = \frac{\alpha}{2\sin\alpha\pi}$). But at $\alpha = 1$ we have the standard relation (1) since $\sin\alpha\pi \to 0$ as $\alpha \to 1$. This is because the nasty point $\varphi = -\pi$ is moved away to infinity and the dispersion of $\tilde{x}$ gains infinite values. The single point $\alpha = 1$ does not have any corresponding value of $\tilde{x}$. Taking this point out from the manifold changes the topology and uncertainty relations. But it changes nothing for smooth functions: this coordinate is good enough because we are short of one point only.

Now we have to discuss the domains of operators under consideration. If $\alpha < 1$, all the functions are defined on a finite interval. If we demand them to be equal to zero at the ends of this interval then the momentum operator would be symmetric but not self-adjoint. It admits an infinite number of self-adjoint extensions but only one of them, $\Psi(-\pi) = \Psi(\pi)$, ensures the finiteness of energy. We use this extension from the very beginning because it is natural for quantum mechanics on the circle.

In the limit of $\alpha = 1$ $\hat{p}_{\tilde{x}}$ is self-adjoint but its domain contains wavefunctions of infinite energy states (those with a discontinuity at $\varphi = \pm\pi$). Still, it can result only in some additional unphysical states for which uncertainty relations are valid. It is more important that $\hat{\tilde{x}}$ with $\alpha = 1$ *does not admit some physical states* (those with $\int_{-\pi}^{\pi} |\Psi(\varphi)|^2 \tan^2\frac{\varphi}{2}\,\mathrm{d}\varphi = \infty$). In principle, it is not a problem due to the following facts: (1) the matrix element (7) is correctly defined for all the physical states and depends on them smoothly; (2) in the L^2 metrics any function $\Psi(\varphi)$ can be regarded as a limit of functions $\Psi_\omega(\varphi) = \Psi(\varphi)\omega(\varphi)$ with $\omega(\varphi)$ equal to unity everywhere apart from a small region around $\varphi = \pm\pi$ and tending to zero fast enough while approaching this point; (3) the momentum dispersion for $|\Psi_\omega\rangle$ tends to that for $|\Psi\rangle$ when $\omega(\varphi) \to 1$ due to the $\cos\varphi/2$ factors in the $\hat{p}_{\tilde{x}}$ definition. But nevertheless, this problem exists for spheres of any dimensionality and it is one more reason to search for better coordinates, which is done in the next section.

5. Coordinates on a sphere

One of the problems of quantum mechanics on a sphere is a proper choice of canonical variables. Spherical coordinates do not solve the problem because they do not provide self-adjoint momenta. Stereographic coordinates are better but they are infinite in the north pole which is just an ordinary point and their operators have too small domains, which results in (1). We overcome this difficulty by 'wrapping' coordinate lines from the projection plane onto the sphere. In the case of two dimensions one can write

$$\begin{cases} \eta = 2\,\mathrm{arccot}\left(\cot\dfrac{\vartheta}{2}\cos\varphi\right) \\ \xi = 2\,\mathrm{arccot}\left(\cot\dfrac{\vartheta}{2}\sin\varphi\right). \end{cases} \tag{8}$$

Actually, this can be easily generalized to any other dimensionality. Lines of constant η or ξ are loops 'hanging down' from the north pole. New coordinates have finite ranges, $\eta \in [0, 2\pi)$, $\xi \in [0, 2\pi)$, hence we do not encounter the problems of the $\alpha = 1$ case of section 4, but the appropriate Hilbert space contains only 2π-periodic functions with respect to both η and ξ.

One can check that thus we have the orthogonal coordinate system on the sphere with closed coordinate lines and one singular point, the north pole. The length element and the metric tensor determinant are

$$\mathrm{d}l^2 = R^2\left(\sin^2\frac{\vartheta}{2} + \cos^2\frac{\vartheta}{2}\cos^2\varphi\right)\mathrm{d}\eta^2 + R^2\left(\sin^2\frac{\vartheta}{2} + \cos^2\frac{\vartheta}{2}\sin^2\varphi\right)\mathrm{d}\xi^2$$

$$g = R^4\left(\sin^2\frac{\vartheta}{2} + \cos^2\frac{\vartheta}{2}\cos^2\varphi\right)^2\left(\sin^2\frac{\vartheta}{2} + \cos^2\frac{\vartheta}{2}\sin^2\varphi\right)^2.$$

In these formulae ϑ and φ should be regarded as functions of new coordinates, η and ξ. The surface element is $\mathrm{d}S = g^{1/2}(\eta, \xi)\,\mathrm{d}\eta\,\mathrm{d}\xi$. The momentum operators

$$\begin{cases} \hat{p}_\eta = \dfrac{\hbar}{\mathrm{i}g^{1/4}} \left(\dfrac{\partial\vartheta}{\partial\eta}\dfrac{\partial}{\partial\vartheta} + \dfrac{\partial\varphi}{\partial\eta}\dfrac{\partial}{\partial\varphi} \right) g^{1/4} \\ \hat{p}_\xi = \dfrac{\hbar}{\mathrm{i}g^{1/4}} \left(\dfrac{\partial\vartheta}{\partial\xi}\dfrac{\partial}{\partial\vartheta} + \dfrac{\partial\varphi}{\partial\xi}\dfrac{\partial}{\partial\varphi} \right) g^{1/4} \end{cases}$$

are written down in terms of old variables, ϑ and φ. Commutational relations

$$[\hat{\eta}, \hat{p}_\eta] = \mathrm{i}\hbar \qquad [\hat{\xi}, \hat{p}_\xi] = \mathrm{i}\hbar \qquad [\hat{p}_\eta, \hat{p}_\xi] = 0$$

can be verified directly but, as in section 2, one has to remember that the left-hand sides of these equations are not defined for the majority of states (it is again a question of periodicity).

One may derive uncertainty relations in two different ways: either using the δ-function approach and a relation analogous to (4) or applying the inequality (3). In any case the result is

$$\Delta\eta \cdot \Delta p_\eta \geqslant 0 \qquad \Delta\xi \cdot \Delta p_\xi \geqslant 0.$$

It is saturated by

$$\Psi_k(\vartheta, \varphi) = \frac{1}{Ag^{1/4}} \exp\left(\frac{2\mathrm{i}k_1}{\hbar} \operatorname{arccot}\left(\cot\frac{\vartheta}{2}\cos\varphi \right) + \frac{2\mathrm{i}k_2}{\hbar} \operatorname{arccot}\left(\cot\frac{\vartheta}{2}\sin\varphi \right) \right)$$

A being a normalization constant. Coordinates η and ξ are preferable because they have finite ranges and do not attribute infinite values to finite points. In general, if any two points separated by a finite distance have a finite difference in their coordinates and all the distances on the manifold are limited by some constant then all coordinates will have finite domains. In particular, every manifold with a positive curvature greater than some positive constant is of this type [13]. If, in addition, all coordinate lines are closed, the inequality (2) is obtained.

6. Topological invariance

If relation (2) holds for some manifold with closed coordinate lines on it, then *it holds also for any manifold diffeomorphic to the initial one* with a coordinate system being moved from one manifold to another by relative diffeomorphism. Disregarding for a moment the smoothness properties, one can speak about homeomorphisms instead of diffeomorphisms and about topological invariance.

Indeed, the inequality (2) is valid for a coordinate system with finite coordinates and closed coordinate lines. It is not changed under any diffeomorphism, hence we only have to obtain relation (2) in the general case. Actually, we have already shown that for two-dimensional manifolds; now we shall do it for an arbitrary dimensionality and in some more detail.

Consider a manifold M^n with a global orthogonal coordinate system. Let one of the coordinates be x, and all the others y. Suppose we have a finite domain and closed coordinate lines for x, $x \in [0, A(y))$. Define a self-adjoint momentum operator

$$\hat{p}_x = \frac{\hbar}{\mathrm{i}} \frac{1}{\sqrt[4]{g(x, y)}} \frac{\partial}{\partial x} \sqrt[4]{g(x, y)}$$

where $g(x, y)$ is the metric tensor determinant for the manifold. Then the rhs of (3) yields

$$\begin{aligned}
\mathrm{Im}\frac{\hbar}{\mathrm{i}}\int_{M^n} &\mathrm{d}x\,\mathrm{d}y\sqrt[4]{g(x,y)}x\Psi(x,y)\frac{\partial}{\partial x}\sqrt[4]{g(x,y)}\Psi^*(x,y)\\
&= \frac{\hbar}{2}\int_{M^n}\mathrm{d}x\,\mathrm{d}y\left(\sqrt{g(x,y)}|\Psi(x,y)|^2 - \frac{\partial}{\partial x}(\sqrt{g(x,y)}x|\Psi(x,y)|^2)\right)\\
&= \frac{\hbar}{2}\left(1 - \int_{M^n\cap\{x=0\}}\mathrm{d}y\,\sqrt{g(0,y)}A(y)|\Psi(0,y)|^2\right)
\end{aligned}$$

the normalization to unity is taken into account. The result equals zero for functions

$$\Psi_k(x,y) = \frac{\exp\left(\frac{2\mathrm{i}\pi kx}{\hbar A(y)}\right)}{\sqrt{\int_{M^n\cap\{x=0\}}\mathrm{d}yA(y)\sqrt[4]{g(x,y)}}}.$$

Again we have the uncertainty relation (2): $\Delta x\cdot\Delta p_x \geqslant 0$. It does not depend upon $g(x,y)$ and $A(y)$, hence it *is invariant under diffeomorphisms* of M^n, as they do not change the topological structure, but only vary the functions mentioned above. The proper coordinate systems can be found at least on the sphere (formulae (8)) and on the torus (obvious, since topologically $\mathfrak{T}^2 \cong S^1\times S^1$), hence the result (2) is valid for any manifold diffeomorphic to the sphere or to the torus.

In general, the problem is to prove the existence of such a coordinate system. We need it in order to define the operators under consideration but not all manifolds possess such systems. Nevertheless, a coordinate can be defined if there is a smooth vector field on a manifold with closed integral curves of finite length and not taking zero values except from the numerable set of singular points $\{t_m\}$, $m = 1, 2, 3\ldots$. Any manifold homeomorphic to a direct product of a circle and an arbitrary manifold, regardless of how complex it is, serves as an example. It has a vector field without singular points generated by rotations of the circle. The vector field defines coordinate and momentum operators, even if the global coordinate system does not exist. In the appendix we show it in the general case.

7. Smooth manifolds and phase spaces

For a free particle on a smooth manifold M^n, its phase space is the tangent bundle TM^n. This bundle is always locally trivial [16] and within any given map allows us to define n momenta (one momentum for each coordinate). But in general it cannot be made uniformly for the whole manifold because the manifold neither has a global coordinate system nor is the bundle globally trivial and admits a smooth section.

The above consideration is appropriate for the following two simplest cases:

(a) The manifold M^n admits a global coordinate system, possibly, with closed coordinate lines, but without singular points. It means that M^n possesses a complete set of independent smooth vector fields (complete parallelizability; Euclidean spaces $\mathbb{R}^n$, tori $\mathfrak{T}^n \cong S^1\times\cdots\times S^1$, cylinders $S^1\times\mathbb{R}^n$, one-, three- and seven-dimensional spheres S^1, S^3, S^7, etc). In this case the tangent bundle is trivial and one can define global coordinates and momenta for the whole M^n.

(b) The manifold M^n admits a global coordinate system with a numerable set (actually, in the previous sections it was a one-element set) of singular points (for example, $S^n\ \forall n$). In this case TM^n becomes trivial after taking away these points from the manifold. Hence one can define global coordinates and momenta everywhere except for the numerable set of points which means nothing for the smooth function properties. In section 5 we have shown how it works.

If the TM^n structure is not so easy the situation is more intricate. One cannot introduce a global coordinate system and the problem of coordinate–momentum uncertainty relations may become senseless because the operators under consideration simply do not exist. Still sometimes it is possible to define m coordinates and momenta, $m < n$, even in this case. For example, in the appendix a manifold with one coordinate common to all the maps is investigated. Loosely speaking, TM^n behaves as a trivial bundle with respect to the common coordinate. A tangent space can be moved along the coordinate line and its initial position is restored after completing a revolution.

In the general case global coordinates are no longer defined and one has to use other observables.

8. Conclusion

Thus, for finite coordinates with closed coordinate lines the uncertainty relation has the form $\Delta x \cdot \Delta p_x \geqslant 0$, and it cannot be made stronger. This is obvious for the eigenstates of the momentum operator. We have shown above how it can be reconciled with the canonical commutational relations. It is worth stressing that one has to make a good choice of the coordinate system on a manifold for the self-adjoint momenta to be well defined.

The uncertainty relation problem for the phase and the number of photons in quantum optics is not directly related to the problem for $\hat{p}_\varphi$ and $\hat{\varphi}$. Creation and annihilation operators $\hat{a}^+$, $\hat{a}$ are not normal ones (they do not commute). Hence, they do not have a decomposition $\hat{a} = \sqrt{\hat{a}\hat{a}^+}\hat{U} = \hat{U}\sqrt{\hat{a}\hat{a}^+}$ with a unitary operator $\hat{U}$. The problem is to define the phase operator correctly [14, 15].

Appendix

Consider a manifold M^n possessing a smooth vector field with closed integral curves of finite length and a hypersurface orthogonal to the curves. Let us attribute the zero value of the first coordinate ($x_1 = 0$) to the latter. The hypersurface points (η) parametrize the set of integral curves. The first coordinate changes along the curves and can be made equal to the path length along the curve from the zero surface in the fixed direction: $x_1 \in [0, A(\eta))$, with $A(\eta)$ being the integral curve length.

Let $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ be a set of maps on the manifold with U_α in M^n and $\phi_\alpha(U_\alpha)$ in Euclidean space $\mathbb{R}^n$. Choose any $\epsilon > 0$ and cut off balls $\mathbb{B}_{\epsilon^m}(t_m)$ of radii ϵ^m (ϵ is powered by m for the total cut-off volume to tend to zero as $\epsilon \to 0$) around the numerable set of singular points and also cut off a vicinity of the $x_1 = 0$ surface:

$$\tilde{M}^n{}_\epsilon = M^n \Bigg\backslash \left(\{x_1 \geqslant A\,(1-\epsilon)\} \cup \{x_1 \leqslant \epsilon A\} \cup \left(\bigcup_m \mathbb{B}_{\epsilon^m}(t_m) \right) \right)$$

with new maps $\tilde{U}_\alpha = U_\alpha \cap \tilde{M}^n_\epsilon$ and $\tilde{\phi}_\alpha = \phi_\alpha|_{\tilde{U}_\alpha}$. The vector field can be moved [17] from $\tilde{U}_\alpha$ to $\tilde{\phi}_\alpha(\tilde{U}_\alpha)$ and the first coordinate x_1 in the map can be put equal to the one introduced invariantly above: $(\tilde{\phi}_\alpha(\zeta))_1 = x_1(\zeta), \forall \zeta \in \tilde{U}_\alpha$. Now, let us choose an orthogonal coordinate system in every set $\tilde{\phi}_\alpha(\tilde{U}_\alpha)$ with this first coordinate $x_1(\zeta)$. The corresponding operator can be defined invariantly and in maps as follows:

$$\hat{s}\Psi(\zeta) = x_1(\zeta)\Psi(\zeta) = (\tilde{\phi}_\alpha(\zeta))_1 \Psi'_\alpha(\tilde{\phi}_\alpha(\zeta)) \qquad \forall \zeta \in \tilde{U}_\alpha$$

where $\Psi'_\alpha = \Psi \circ \tilde{\phi}_\alpha^{-1}$.

The measure can be written as $\mathrm{d}\mu = \left(\prod_i \mathrm{d}x_i^\alpha\right) g_\alpha^{1/2}\left(x_i^\alpha\right)$, where x_i^α are coordinates in $\tilde{\phi}_\alpha(\tilde{U}_\alpha)$. One can introduce a momentum operator in this map $\hat{p}_s^\alpha = \frac{\hbar}{\mathrm{i}} \frac{1}{g^{1/4}} \frac{\partial}{\partial x_1} g^{1/4}$ which generates a shift along an integral curve of the vector field and is invariant under any change of map (the Jacobian does not depend on x_1). It can be defined uniformly for the whole manifold $\hat{p}_s\Psi(\zeta) = \sum_\alpha \chi_\alpha(\zeta)\hat{p}_s^\alpha\Psi(\zeta)$ with χ_α being a smooth partition of unity for $\{\tilde{U}_\alpha, \tilde{\phi}_\alpha\}$.

In the matrix element

$$\langle\Psi_1|\hat{p}_s\Psi_2\rangle = \lim_{\epsilon\to 0}\int_{\tilde{M}^n_\epsilon} \mathrm{d}\mu\, \Psi_1^*\hat{p}_s\Psi_2$$

one can integrate by parts and all nonintegral terms vanish at the edges of maps with the partition of unity functions χ_α except of cut-offs where they cancel each other when $\epsilon \to 0$. Therefore

$$\langle\Psi_1|\hat{p}_s\Psi_2\rangle = \langle\hat{p}_s\Psi_1|\Psi_2\rangle - \frac{\hbar}{\mathrm{i}}\sum_\alpha\int \mathrm{d}\mu\, \frac{\partial\chi_\alpha}{\partial x_1}\Psi_1^*\Psi_2 = \langle\hat{p}_s\Psi_1|\Psi_2\rangle$$

because $\sum_\alpha \chi_\alpha \equiv 1$. We have proved that $\hat{p}_s$ is self-adjoint.

Consider now $\langle\hat{s}\Psi|\hat{p}_s\Psi\rangle$. If one integrates it by parts, the cancellation along the cut-off $x_1 = 0$ does not occur because the coordinate x_1 is not periodic. Let α^+ be such α that $\tilde{U}_\alpha$ touches the edge $x_1 = A(1-\epsilon)$, and α^- be such α that $\tilde{U}_\alpha$ touches the edge $x_1 = \epsilon A$. Then (with the normalization condition):

$$\begin{aligned}\frac{2}{\hbar}\mathrm{Im}\langle\hat{s}\Psi|\hat{p}_s\Psi\rangle &= \sum_\alpha\int_{\tilde{M}^n_\epsilon} \mathrm{d}x_1^\alpha\cdots\mathrm{d}x_n^\alpha g_\alpha^{1/2}(x^\alpha)|\Psi_\alpha(x^\alpha)|^2\\ &\quad-\sum_{\alpha^+}\int_{\tilde{M}^n_\epsilon\cap\{x_1=A(1-\epsilon)\}} \mathrm{d}x_2^{\alpha^+}\cdots\mathrm{d}x_n^{\alpha^+} A(1-\epsilon)g_{\alpha^+}^{1/2}\big(x^{\alpha^+}\big)\big|\Psi_{\alpha^+}\big(x^{\alpha^+}\big)\big|^2\\ &\quad+\sum_{\alpha^-}\int_{\tilde{M}^n_\epsilon\cap\{x_1=\epsilon A\}} \mathrm{d}x_2^{\alpha^-}\cdots\mathrm{d}x_n^{\alpha^-}\epsilon A g_{\alpha^-}^{1/2}\big(x^{\alpha^-}\big)\big|\Psi_{\alpha^-}\big(x^{\alpha^-}\big)\big|^2\\ &\underset{\epsilon\to 0}{\longrightarrow} 1 - \int \mathrm{d}\mu\,\delta(x_1)A|\Psi|^2\end{aligned}$$

where $\mathrm{d}\mu\,\delta(x_1)$ is a measure at the hypersurface $x_1 = 0$ induced by the Riemannian structure of $\tilde{M}^n_\epsilon$.

By using (3) one can get the uncertainty relation of the form (2): $\Delta s \cdot \Delta p_s \geqslant 0$. It is saturated by states which in the map $\tilde{U}_\alpha$ can be written as

$$\frac{\exp\left(\frac{2\mathrm{i}\pi k x_1^\alpha}{\hbar A}\right)}{g_\alpha^{1/4}(x^\alpha)} f_\alpha\big(x_2^\alpha,\ldots,x_n^\alpha\big)$$

with functions f_α providing invariance under any change of map.

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